# A multivariate regime-switching mean reverting process and its application to the valuation of credit risk

Yinghui Dong[a,b] Kam C. Yuen[c] Chongfeng Wu[b]

[a] *Department of Mathematics and Physics, Suzhou University of Science and Technology Suzhou 215009, P. R. China*

[b] *Financial Engineering Research Center, Shanghai Jiao Tong University Shanghai 200052, P. R. China*

[c] *Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong*

**Abstract** In this paper, we study the counterparty risk on a credit default swap (CDS) and the valuation of a first-to-default basket swap on three underlyings under a common shock model with regime-switching intensities. We assume that the defaults of all the names are driven by some shock events, whose arrivals are governed by a multivariate regime-switching shot noise process. Based on some expressions for the joint Laplace transform of the regime-switching shot noise processes, we give explicit formulas for the spread of the CDS contract with and without counterparty risk and the spread of the first-to-default basket swap on the three underlyings.

**Key words:** credit default swap, counterparty risk, common shock, multivariate regime-switching shot noise process, first-to-default basket swap

AMS2000 classification: 91B25;60J27;60G55.

## 1. Introduction

Counterparty credit risk is the risk that the counterparty to a financial contract will default prior to the expiration of the contract and will not make all the payments required by the contract. Once two counterparties enter into a financial transaction, they should take credit risk against each other. To value counterparty credit risk, the most important task is to model the default dependence among the counterparties. Credit default swaps are the most widely traded form of credit derivative. A credit default swap (CDS) is a financial swap agreement between the buyer of the default protection on a reference risky entity and the seller of the default protection. The protection seller receives fixed periodic payments (CDS premium) from the protection buyer, in return for compensating the buyer's losses on the reference entity when a credit event occurs. This paper focuses on valuing a single-name CDS with and without counterparty risk as well as a first-to-default CDS on three underlyings.

The reduced-form approach is one of the most popular methods to model the default correlation. Reduced-form models, introduced by Duffie and Singleton [12], Jarrow and Turnbull [20], and others, focus directly on the modeling of the default probability. This methodology does not intend to explain the default of a firm by means of an economic construction. Instead, the time of default is defined as the first jump time of a point process. There exist four major approaches to introduce default correlation within the reduced-form framework: the conditionally independent approach, the copula approach, the default contagion models, and the common shock models. In the conditionally independent default models, one may set the default intensities of the firms in the portfolio to be driven by a common set of macro-economic factors. Therefore, conditional on the realization of the macro-economic state variables, the default times are mutually independent; see, for example, Duffie and Gârleanu [11] and Graziano and Rogers [9]. In the copula models, the dependence structure is linked through a copula function; see, for example, Schonbucher and Schubert [24] and Hull and White [19]. Default contagion is another approach to model the default correlation. The contagion models study the direct interaction of firms in which the default probability of one firm may change upon defaults of some other firms in the portfolio; see, for example, Davis and Lo [8] and Ma and Yun [23]. The common shock models are based on the idea that a firm's default is driven by exogenous events, for example, policy events, natural catastrophes events, etc. Therefore, simultaneous defaults may occur under the common shock models; see, for example, Lindskog and McNeil [22], Giesecke [16], Brigo et al. [3], and Bielecki et al. [1].

This paper focuses on a model with common shock. In the framework of common shock models, Lindskog and McNeil [22] and Giesecke [16] both assume the shock events arrive as independent Poisson processes. In this paper, we extend their work to the case that the arrival intensities of the external shock events are modeled by some conditionally independent Cox processes. Since the default intensities of the firms can be expressed as some linear combination of the arrival intensities of the shock events under some suitable assumptions within the common shock framework, the default dependence in our model is not only due to common shock, but also the dependence among the arrival intensities. In the literature, the shot noise processes are good tools for describing the arrival intensities as they allow for explicit solutions to many important quantities in derivative pricing. For example, Gaspar and Schmidt [15] consider a multivariate default model driven by the shot noise processes and show that the shot noise processes can describe historical data very well and give a better fit in calibration than the affine jump-diffusion models proposed by Duffie and Gârleanu [11]. Also, Cox and Isham [5] and Dassios and Jang [7] show that the shot noise processes can be used to measure the impact of major events on the intensities. However, most existing works on the shot noise models assume that the jumps are driven by a compound Poisson process. Intuitively, the jumps should be related to the macro-economic conditions since we have witnessed that recent global financial crises do have significant impact on international financial markets, in particular on the values of credit derivatives. In fact, default risk is influenced very much by business cycles or macro-economy. For instance, default risk typically declines during economic expansion because strong earnings keep overall defaults rates low; and it increases during economic recession because weak earnings make it more difficult to repay loans or bond payments. In view of these, there is a practical need to develop some credit risk models, which can take into account changes in market regimes.

Markov regime-switching models have been used by a lot of research in different branches of modern financial economics to capture changes in market regimes. For example, see Buffington and Elliott [4], Yuen and Yang [26], Shen and Siu [25], Dong et al. [10], and Elliott and Siu [13]. Regime switches are often interpreted as structural changes in macro-economic conditions and in different stages of business cycles. The advantages of using Markov regime-switching models have been empirically verified in various financial markets. For example, in the stock market, by using monthly returns data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices, Hardy [18] finds that the regime-switching lognormal model fits to the monthly returns data much better than other econometric models such as the independent lognormal model and the ARCH type models. In the credit market, by an empirical analysis of the corporate bond market over the course of the last 150 years, Giesecke et al. [17] point out that there exist three regimes, associated with high, middle, and low default risk.

Motivated by Gaspar and Schmidt [15], Buffington and Elliott [4], and Giesecke et al. [17], we propose a multivariate regime-switching shot noise process to model the arrival intensities of the shock events. This paper aims at providing a flexible and tractable model for correlated defaults which take into account the changes in market regimes or environments due to financial crises. The contribution of this article is to provide a correlated default model with regime-switching intensities, in which the default intensities of all parties can change simultaneously over time depending on the state of the underlying Markov chain. Furthermore, the model leads to analytic formulas for the CDS spreads with and without counterparty risk. The rest of the paper is organized as follows. In section 2, we introduce a common shock model, in which the shock events arrive as some dependent regime-switching shot noise processes. We also present some preliminary results in this section. Section 3 derives the joint Laplace transform of the shot noise processes. Based on the joint Laplace transform, we obtain the joint survival distributions. Section 4 incorporates the common shock model into the Markov copula framework, which was considered by Crépy et al. [6] and Bielecki et al. [1]. Section 5 gives the closed-form formulas for the CDS spreads with and without counterparty risk and the spread of the first-to-default basket swap on three underlyings. Section 6 presents some numerical results. Finally, Section 7 concludes the paper.

## 2. Default dependence and some preliminary results

In this section, we model the dependence structure within the common shock framework under a Markov environment. Consider a continuous-time model with a finite time horizon $[0, T]$ with $T < \infty$. Let $\{\Omega, \Im, \{\Im_t\}_{0\leq t\leq T}, P\}$ be a filtered complete probability space, where $P$ is the risk neutral measure and $\{\Im_t\}_{0\leq t\leq T}$ is a filtration satisfying the usual conditions of right continuity and completeness. Throughout the paper, it is assumed that all random variables are well defined on this probability space and $\Im_T-$measurable.

Denote by $D(t_1, t_2)$ the stochastic factor giving the discounted value of one at time $t_1$ due at time $t_2$. Consider a CDS contract with notional value of one, continuous spread rate payments, and maturity time $T$. Indices 0, 1, 2 refer to quantities related to the investor, the reference entity, and the counterparty. Also, denote by $\tau_0, \tau_1$ and $\tau_2$ be the default times of the investor, the reference entity and the counterparty, respectively; and denote by $R_1$ the recovery of the reference entity which is supposed to be a constant. In this paper, it is assumed that all the cash flows and prices are considered from the perspective of the investor.

In order to derive an explicit expression for the fair spread of a CDS contract, we construct a default dependence structure in the reduced-form framework. Motivated by Lindskog and McNeil [22], we assume that there are $m$ event types called factor names, which can generate events potentially causing joint defaults. We further assume that there exists a process $\{X_t\}_{t\geq 0}$ which governs the dynamics of all arrivals of the $m$ types of shock events and the interest rate. Here, the process $\{X_t\}_{t\geq 0}$ is a homogeneous Markov chain with generator $Q = (q_{ij})$ describing the macro-economic conditions. The state space of $X$ can be taken to be, without loss of generality, the set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^* \in R^N$ with the symbol $*$ denoting the transpose of a vector or a matrix. Denote the corresponding filtration by $\Im_t^X = \sigma(X_u, 0 \leq u \leq t)$. Elliott et al. [14] provide the following semi-martingale decomposition for $X_t$:

$$dX_t = Q^* X_t dt + dM_t, \tag{2.1}$$

where $M_t$ is an $\Im_t^X$-martingale.

Let $\langle ., . \rangle$ denote a scalar product in $R^N$, that is, for any $\mathbf{x}, \mathbf{y} \in R^N$, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^N x_i y_i$. Assume that the discount factor is given by $D(0, t) = \exp\{-\int_0^t r_s ds\}$, where the interest rate $r_t$ has the form

$$r_t = r_0 + \int_0^t h_0(t-s) dJ_s^0 \doteq r_0 + L_t^0. \tag{2.2}$$

Here, $r_0 = \langle \mathbf{r}, X_0 \rangle$, where $\mathbf{r} = (r_1, r_2, \cdots, r_N)^* \in R^N$ with $r_i > 0$, for each $i = 1, 2, \cdots, N$; $h_0(.)$ is an $R^+$-valued deterministic function; and $J_t^0 = \sum_{i=1}^{M^0(t)} Y_j^0$ is a regime-switching compound Poisson process with $M^0(t)$ being a regime-switching Poisson process. Write the intensity of $M^0(t)$ as $\mu_0(s) = \langle \boldsymbol{\mu_0}, X_s \rangle$, for a positive vector $\boldsymbol{\mu_0} = (\mu_0^1, \cdots, \mu_0^N)^*$. That is, if $X_s = e_j$ for all $s$ in a small interval $(t, t+h]$, then $M^0(t+h) - M^0(t)$ has a Poisson distribution with parameter $\mu_0^j$. Assume that, given the path of the Markov chain $X$, $\{Y_1^0, Y_2^0, \cdots\}$ is a sequence of independent and identically distributed with conditional density $f_t^0$ concentrated on $(0, \infty)$ and independent of $M^0(t)$, where $f_t^0(.) = \langle \mathbf{f}^0(.), X_t \rangle$ for a vector $\mathbf{f}^0(.) = (f^{01}(.), \cdots, f^{0N}(.))^*$. Furthermore, it is assumed that $M^0(t)$ does not jump at the jump times of Markov chain $X$. Since jumps of interest rate are possibly due to some extraordinary market events, such as market crashes and interventions of central banks or monetary authorities, the regime-switching shot noise process can be used to describe the impact of major market events on the movement of interest rate. If there is no regime-switching, the process $L_t^0$ is called a shot noise process, which was studied in Gaspar and Schmidt [15]. Here, we extend the study of shot noise process to the case with regime switching.

For the arrivals of shock events, we assume that the $m$ types of shock events arrive as Cox processes $\{N^1(t), t \geq 0\}, \cdots, \{N^m(t), t \geq 0\}$ with stochastic intensities $\lambda_t^1, \cdots, \lambda_t^m$ generating a filtration $\Im_t^{\boldsymbol{\lambda}} = \Im_t^{\lambda^1} \vee \Im_t^{\lambda^2} \vee \cdots \vee \Im_t^{\lambda^m}$, with $\Im_t^{\lambda^i} = \sigma(\lambda_s^i, 0 \leq s \leq t)$. Furthermore, given $\Im_t^{\boldsymbol{\lambda}}$, it is assumed that

$\{N^1(t), t \geq 0\}, \cdots, \{N^m(t), t \geq 0\}$ are mutually independent. For each $i = 1, \cdots, m$, the intensity $\lambda_t^i$ is given by

$$\lambda_t^i = \lambda_0^i + \int_0^t h_i(t-u) dJ_u^i \doteq \lambda_0^i + L_t^i. \tag{2.3}$$

Here, $\lambda_0^i = \langle \boldsymbol{\lambda}^i, X_0 \rangle$, where $\boldsymbol{\lambda}^i = (\lambda^{i1}, \lambda^{i2}, \cdots, \lambda^{iN})^* \in R^N$ with $\lambda^{ij} > 0$ for $i = 0, 1, 2$, and $j = 1, \cdots, N$; $h_i(.)$ is an $R^+$-valued deterministic function; and $J_t^i = \sum_{j=1}^{M^i(t)+M^0(t)} Y_j^i$ is a regime-switching compound Poisson process, where $M^0(t)$ is given in (2.2), and $M^i(t)$ is also a regime-switching Poisson process with intensity $\mu_i(s) = \langle \boldsymbol{\mu}_i, X_s \rangle$ for a positive vector $\boldsymbol{\mu}_i = (\mu_i^1, \cdots, \mu_i^N)^*$; $Y_n^i$ is the size of the $n$th jump that is independent of $M^i(t)$ given the state of the Markov chain. Given $\Im_t^X$, it is assumed that $M^0(t), M^1(t) \cdots, M^m(t)$ are mutually independent, and that $\{Y_j^0, j = 1, 2, \cdots\}, \cdots, \{Y_j^m, j = 1, 2, \cdots\}$ are mutually independent and independent of $M^0(t), \cdots, M^m(t)$. Furthermore, given the path of the Markov chain $X$, we assume that for each $i = 1, 2, \cdots, m$, the jump sizes $Y_j^i$, $j = 1, 2, \cdots$ have a common conditional density $f_t^i$ concentrated on $(0, \infty)$, where $f_t^i(.) = \langle \mathbf{f}^i(.), X_t \rangle$, with $\mathbf{f}^i(.) = (f^{i1}(.), \cdots, f^{iN}(.))^*$. Note that the stochastic interest rate and the intensities $\lambda_t^1, \cdots, \lambda_t^m$ given by (2.3) are driven by a multivariate regime-switching shot noise process with common jumps. Intuitively, extraordinary market events which trigger jumps in the interest rate may also trigger simultaneous jumps in the arrival intensities of the shock events. Therefore, all the intensities are influenced by $M^0(t)$ which counts the number of the arrivals of extraordinary market events. It follows from (2.3) that the intensity of each group is also influenced by some factors of its group. In particular, $M^i(t)$ counts the number of the factor events occurring in group $i$.

**Remark 2.1.** *For each $i = 0, 1, \cdots, m$, if $h_i(t) = 1$, then $L_t^i$ becomes a regime-switching compound Poisson process; if $h_i(t) = \exp\{-a^i t\}$, where $a^i > 0$ is a constant, then $L_t^i$ is a mean-reverting regime-switching Markov process, and it solves the stochastic differential equation*

$$dL_t^i = -a^i L_t^i dt + dJ_t^i, \quad L_0^i = 0. \tag{2.4}$$

We now construct a default dependence structure by the thinning of the Cox processes $N^e(t), e = 1, \cdots, m$. Define the collection

$$\mathcal{S} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

In order to use the Cox framework to specify the random default times, we define the counting processes $N_i(t)$, $i = 0, 1, 2$, and $N_s(t)$, $s \in \mathcal{S}$, which count shocks in the interval $(0, t]$ resulting in default of name $i$ and simultaneous defaults of the names in $s$, respectively. By definition,

$$N_i(t) = \sum\nolimits_{s \in \mathcal{S}, i \in s} N_s(t).$$

For $k = 1, \cdots, m$ and $i = 0, 1, 2$, let $p_{ki}(t)$ be the conditional probability of finding the $i$th credit name defaulted knowing that factor name $k$ has generated an event during $(t, t + dt)$. Suppose that each event of $k$th type occurring at time $t$ has probability $p_{ki}(t)$ of generating a default of firm $i$ only, and that defaults of the credit names are conditionally independent given that an event of a certain type has arrived. These imply that the conditional probability density that only the names in $s \in \mathcal{S}$ defaulting during $(t, t + dt)$ is $\sum_{k=1}^m \lambda_t^k \prod_{i \in s} p_{ki}(t) \prod_{j \notin s} \overline{p}_{kj}(t)$, where $\overline{p}_{kj}(t) = 1 - p_{kj}(t)$. For simplicity, we further assume that $p_{kj}(t) \equiv p_{kj}$ for $k = 1, \cdots, m$ and $j = 0, 1, 2$. Then, it follows from the above assumptions that the processes $\{N_s(t), t \geq 0\}, s \in \mathcal{S}$, are independent given $\Im_t^{\boldsymbol{\lambda}}$, and that the intensity of the counting process $N_s(t)$ is given by

$$q_s(t) = \sum\nolimits_{k=1}^m \lambda_t^k \prod\nolimits_{i \in s} p_{ki} \prod\nolimits_{j \notin s} \overline{p}_{kj}.$$

It is easy to check that

$$\sum_{s \in \mathcal{S}} q_s(t) = \sum_{k=1}^m \lambda_t^k (1 - \prod_{l=0}^2 \overline{p}_{kl}),$$

and that the intensity of $N_i(t)$ has the form

$$q_i(t) = \sum_{s\in\mathcal{S}, i\in s} q_s(t) = \sum_{k=1}^{m} \lambda_t^k p_{ki}.$$

**Remark 2.2.** *As was pointed out in Lindskog and McNeil [21], the process $N_j(t)$ represents the number of defaults of name $j$ in the interval $(0, t]$. Since CDS contracts end at the time of the first default, we focus on the first jump of the process $N_j(t)$.*

Denote the filtration by $\Im_t^L = \Im_t^{L^0} \vee \Im_t^{L^1} \vee \cdots \vee \Im_t^{L^m}$, where $\Im_t^{L^i} = \sigma(L_u^i : 0 \le u \le t), i = 0, 1, \cdots, m$. With $N_i(t)$, we define the default time of name $i$ as

$$\tau_i = \inf\{t \ge 0 : N_i(t) = 1\}, i = 0, 1, 2.$$

Therefore, we have

$$P(\tau_i > t | \Im_t^L \vee \Im_t^X) = P(N_i(t) = 0 | \Im_t^L \vee \Im_t^X) = e^{-\int_0^t q_i(u)du}.$$

Define the default processes as

$$H_t^i = 1_{\{\tau_i \le t\}}, \qquad i = 0, 1, 2,$$

and denote the filtration by

$$\Im_t = \Im_t^X \vee \Im_t^L \vee \Im_t^0 \vee \Im_t^1 \vee \Im_t^2,$$

where $\Im_t^i = \sigma(H_u^i : 0 \le u \le t), i = 0, 1, 2$.

In order to derive the joint survival distribution, we need to use the first jump time of the counting process $N_s(t)$ given by

$$\tau_s = \inf\{t \ge 0 : N_s(t) = 1\}, \qquad s \in \mathcal{S}.$$

For example, if $s = \{1, 2\}$, then $N_s(t)$ counts shocks which cause simultaneous defaults of names 1 and 2, but not of name 0. Since $\{N_s(t), t \ge 0\}$, for $s \in \mathcal{S}$, are conditionally independent Cox processes, the stopping times $\tau_s$, for $s \in \mathcal{S}$, are also conditionally independent. Furthermore, the conditional and unconditional distributions for $\tau_s$ can be expressed as

$$P(\tau_s > t | \Im_t^L \vee \Im_t^X) = P(N_s(t) = 0 | \Im_t^L \vee \Im_t^X) = e^{-\int_0^t q_s(u)du}. \tag{2.5}$$

Then, it follows from the relationship between $N_i(t)$ and $N_s(t)$ that the default time $\tau_i$ can be rewritten as

$$\tau_i = \inf\{t \ge 0 : \sum_{s\in\mathcal{S}: i\in s} N_s(t) = 1\} = \min_{s\in\mathcal{S}: i\in s} \tau_s, \qquad i = 0, 1, 2. \tag{2.6}$$

The next proposition gives the conditional joint survival probability of the three firms.

**Proposition 2.1.** *For $t_i \ge 0, i = 0, 1, 2$, we have*

$$\begin{aligned} & P(\tau_0 > t_0, \tau_1 > t_1, \tau_2 > t_2 | \Im_{t_{(2)}}^L \vee \Im_{t_{(2)}}^X) \\ = \; & e^{-\int_0^{t_{(0)}} \sum_{k=1}^{m} \lambda_u^k (1 - \prod_{l=0}^{2} \bar{p}_{kl}) du - \int_{t_{(0)}}^{t_{(1)}} \sum_{k=1}^{m} \lambda_u^k (1 - \prod_{l=(1)}^{(2)} \bar{p}_{kl}) du - \int_{t_{(1)}}^{t_{(2)}} \sum_{k=1}^{m} \lambda_u^k p_{k(2)} du}, \end{aligned} \tag{2.7}$$

*where $t_{(i)}{}'s$ are ordered times with $0 \le t_{(0)} \le t_{(1)} \le t_{(2)}$ and $(i)$ refers to the credit name associated with the $(i+1)$th ordered time.*

**Proof.** Using (2.6), we obtain

$$\begin{aligned}
& P(\tau_0 > t_0, \tau_1 > t_1, \tau_2 > t_2 | \Im^L_{t_{(2)}} \vee \Im^X_{t_{(2)}}) \\
= \; & P(\tau_{(0)} > t_{(0)}, \tau_{(1)} > t_{(1)}, \tau_{(2)} > t_{(2)} | \Im^L_{t_{(2)}} \vee \Im^X_{t_{(2)}}) \\
= \; & P(\tau_{\{(0)\}} > t_{(0)}, \tau_{\{(1)\}} > t_{(1)}, \tau_{\{(0),(1)\}} > t_{(1)}, \min_{(2)\in s} \tau_s > t_{(2)} | \Im^L_{t_{(2)}} \vee \Im^X_{t_{(2)}}) \\
= \; & P(\tau_{\{(0)\}} > t_{(0)} | \Im^L_{t_{(2)}} \vee \Im^X_{t_{(2)}}) \prod_{s=\{(1)\},\{(0,1)\}} P(\tau_s > t_{(1)} | \Im^L_{t_{(2)}} \vee \Im^X_{t_{(2)}}) \\
& \times \prod_{s\in\mathcal{S},(2)\in s} P(\tau_s > t_{(2)} | \Im^L_{t_{(2)}} \vee \Im^X_{t_{(2)}}),
\end{aligned}$$

where the last equality follows from the conditional independence of $\tau_s, s \in \mathcal{S}$. Then, substituting (2.5) into the above equality gives the result. □

The next two results are very useful for deriving the spread of CDS.

**Lemma 2.1.** *For any $\Im^X_T \vee \Im^L_T-$measurable random variable $Y$ and any $u \geq t \geq 0$, we have*

$$E\left[1_{\{\tau_i > u\}} Y | \Im_t\right] = 1_{\{\tau_i > t\}} E\left[Y e^{-\int_t^u q_i(v)dv} | \Im^L_t \vee \Im^X_t\right], i = 0, 1, 2;$$

*for any $s \in \mathcal{S}$,*

$$E\left[1_{\{\tau_s > u\}} Y | \Im_t\right] = 1_{\{\tau_s > t\}} E\left[Y e^{-\int_t^u q_s(v)dv} | \Im^L_t \vee \Im^X_t\right];$$

*and*

$$E\left[1_{\{\min\limits_{s\in\mathcal{S}} \tau_s > u\}} Y | \Im_t\right] = 1_{\{\min\limits_{s\in\mathcal{S}} \tau_s > t\}} E\left[Y e^{-\int_t^u \sum_{k=1}^m \lambda^k_v (1-\prod_{l=0}^2 \overline{p}_{kl})dv} | \Im^L_t \vee \Im^X_t\right].$$

**Proof.** See Corollary 5.1.1 in Bielecki and Rutkowski [2]. □

**Lemma 2.2.** *Let $\overline{\tau} = \min\limits_{s\in\mathcal{S}} \tau_s$. Let $Z$ be a bounded $\Im^X_T \vee \Im^L_T-$predictable process. Then, for any $0 \leq t < \infty$ and $s' \in \mathcal{S}$,*

$$E\left[Z_{\tau_{s'}} 1_{\{\overline{\tau}=\tau_{s'}, t<\tau_{s'}\leq s\}} | \Im_t\right] = 1_{\{\overline{\tau}>t\}} E\left[\int_t^s Z_u q_{s'}(u) e^{-\int_t^u \sum_{k=1}^m \lambda^k_v (1-\prod_{l=0}^2 \overline{p}_{kl})dv} du | \Im^L_t \vee \Im^X_t\right].$$

**Proof.** From Lemma 2.1, we have

$$E\left[Z_{\tau_{s'}} 1_{\{\overline{\tau}=\tau_{s'}, t<\tau_{s'}\leq s\}} | \Im_t\right] = 1_{\{\overline{\tau}>t\}} e^{\int_0^t \sum_{k=1}^m \lambda^k_u (1-\prod_{l=0}^2 \overline{p}_{kl})du} E\left[Z_{\tau_{s'}} 1_{\{\overline{\tau}=\tau_{s'}, t<\tau_{s'}\leq s\}} | \Im^L_t \vee \Im^X_t\right]. \quad (2.8)$$

So, it remains to derive the expression for the conditional expectation in the above equality. We first assume $Z$ is a stepwise $\Im^X_T \vee \Im^L_T-$predictable process, that is, for $t < u \leq s$, $Z_u = \sum_{i=0}^n Z_{t_i} 1_{\{t_i<u\leq t_{i+1}\}}$, where $t_0 = t < t_1 < \cdots < t_{n+1} = s$, and $Z_{t_i}$ is $\Im^L_{t_i} \vee \Im^X_{t_i}-$measurable random variable for $i = 0, 1, \cdots, n$. Then,

$$E\left[Z_{\tau_{s'}} 1_{\{\overline{\tau}=\tau_{s'}, t<\tau_{s'}\leq s\}} | \Im^L_t \vee \Im^X_t\right]$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} E\left[Z_{t_i} E\left[1_{\{t_i<\tau_{s'}\le t_{i+1},\overline{\tau}=\tau_{s'}\}}|\Im_{t_{i+1}}^L \vee \Im_{t_{i+1}}^X\right]|\Im_t^L \vee \Im_t^X\right] \\
&= \sum_{i=0}^{n} E\left[Z_{t_i}\int_{t_i}^{t_{i+1}} q_{s'}(u) e^{-\int_0^u \sum_{k=1}^{m}\lambda_v^k(1-\prod_{l=0}^{2}\overline{p}_{kl})dv} du|\Im_t^L \vee \Im_t^X\right], \qquad (2.9)
\end{aligned}
$$

where the second equality holds since $\tau_s, s \in \mathcal{S}$, are conditionally independent. Define

$$
F(t) = \int_0^t q_{s'}(u) e^{-\int_0^u \sum_{k=1}^{m}\lambda_v^k(1-\prod_{l=0}^{2}\overline{p}_{kl})dv} du.
$$

Then, (2.9) can be rewritten as

$$
\begin{aligned}
E\left[Z_{\tau_{s'}}1_{\{\overline{\tau}=\tau_{s'},t<\tau_{s'}\le s\}}|\Im_t^L \vee \Im_t^X\right] &= \sum_{i=0}^{n} E\left[Z_{t_i}(F(t_{i+1})-F(t_i))|\Im_t^L \vee \Im_t^X\right] \\
= \sum_{i=0}^{n} E\left[\int_{t_i}^{t_{i+1}} Z_u dF(u)|\Im_t^L \vee \Im_t^X\right] &= E\left[\int_t^s Z_u dF(u)|\Im_t^L \vee \Im_t^X\right]. \qquad (2.10)
\end{aligned}
$$

For any $Z$, we can use a suitable sequence of bounded, stepwise, $\Im_T^X \vee \Im_T^L-$predictable processes to approximate $Z$. So, (2.10) holds for any bounded $\Im_T^X \vee \Im_T^L-$predictable process $Z$. Finally, substituting (2.10) into (2.8) yields the result. □

## 3. Laplace transforms and survival distributions

In this section, we give the joint Laplace transform of the regime-switching shot noise processes and the integrated regime-switching shot noise processes under the assumption that $L_t^i$ is a mean-reverting regime-switching process defined in (2.4) for each $i = 0, 1, \cdots, m$. With the joint Laplace transform, we can obtain explicit formulas for the joint survival distributions.

For $c^i \ge 0, d^i \ge 0$, and $i = 0, 1, \cdots, m$, let

$$
V(t,T) = E\left[e^{-\int_t^T \sum_{i=0}^{m} c^i L_s^i ds - \sum_{i=0}^{m} d^i L_T^i}|\Im_t^L \vee \Im_t^X\right],
$$

where $L_t^i, i = 0, 1, \cdots, m$, are given in (2.4). Note that $L_t^i > 0$ for $i = 0, 1, \cdots, m$. Consequently, $V(t,T)$ is a bounded function. Since $(X_t, L_t^0, L_t^1 \cdots, L_t^m)^*$ is an $(m+2)-$dimensional Markov process with respect to $\Im_t^L \vee \Im_t^X$, we have

$$
\begin{aligned}
V(t,T) &= E\left[e^{-\int_t^T \sum_{i=0}^{m} c^i L_s^i ds - \sum_{i=0}^{m} d^i L_T^i}|L_t^i, i = 0, 1, \cdots, m, X_t\right] \\
&=: \theta(t, T, L_t^0, L_t^1, \cdots, L_t^m, X_t).
\end{aligned}
$$

Write

$$
\begin{aligned}
\theta_i = \theta(t, T, L_t^0, L_t^1, \cdots, L_t^m, e_i), i = 1, 2, \cdots, N, \\
\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_N)^* \in R^N.
\end{aligned}
$$

The following result gives the explicit expression for $\theta(t, T, L_t^0, L_t^1, \cdots, L_t^m, X_t)$.

**Theorem 3.1.** *Let* $\mathbf{c} = (c^0, c^1, \cdots, c^m)^* \in \mathbf{R^{m+1}}$ *and* $\mathbf{d} = (d^0, d^1, \cdots, d^m)^* \in \mathbf{R^{m+1}}$ *with* $c^i \ge 0, d^i \ge 0$ *for each* $i = 0, 1, \cdots, m$. *Then, we have*

$$
V(t,T) = e^{-\sum_{i=0}^{m}(c^i\xi^i(t,T)+d^i e^{-a^i(T-t)})L_t^i}\langle\Psi_1(\mathbf{c},\mathbf{d},t,T), X_t\rangle, \qquad (3.1)
$$

*where*

$$\xi^i(t,T) = (1-e^{-a^i(T-t)})/a^i, i=0,1,\cdots,m,$$

*and the* $N-$*dimensional vector* $\Psi_1(\mathbf{c},\mathbf{d},t,T)$ *solves*

$$\frac{\partial \Psi_1}{\partial t} + (Q+\mathbf{F}_t(\mathbf{c},\mathbf{d}))\Psi_1(\mathbf{c},\mathbf{d},t,T) = 0,\ \ \Psi_1(\mathbf{c},\mathbf{d},T,T) = \mathbf{1}, \tag{3.2}$$

*with* $\mathbf{1} = (1,1,\cdots,1)^* \in \mathbf{R^N}$, *and* $\mathbf{F}_s$ *being an* $N$*-dimensional vector with the* $j$*th component given by*

$$F_s^j(\mathbf{c},\mathbf{d}) = \sum_{i=1}^m \mu_i^j(g_s^{ij}(c^i,d^i)-1) + \mu_0^j(\prod_{i=0}^m g_s^{ij}(c^i,d^i)-1),$$

*and*

$$g_s^{ij}(c^i,d^i) = \int_0^\infty e^{-(c^i\xi^i(s,T)+d^ie^{-a^i(T-s)})x}f^{ij}(x)dx, \qquad i=0,1,\cdots,m, \quad j=1,\cdots,N.$$

**Proof.** We use the martingale approach to derive (3.1). Consider the function

$$\overline{V}(t,T) = U_t\theta(t,T,L_t^0,\cdots,L_t^m,X_t),$$

where $U_t = \exp(-\int_0^t \sum_{i=0}^m c^iL_s^ids)$. Applying Itô's differentiation rule to $\overline{V}(t,T)$ yields

$$\begin{aligned}
d\overline{V}(t,T) &= U_t(\frac{\partial}{\partial t} - \sum_{i=0}^m a^iL_t^i\frac{\partial}{\partial L^i} - \sum_{i=0}^m c^iL_t^i)\theta(t,T,L_t^0,L_t^1,\cdots,L_t^m,X_t)dt \\
&+ U_t\sum_{i=1}^m((\theta(t,T,L_t^0,\cdots,L_t^i,\cdots,L_t^m,X_t) - \theta(t,T,L_t^0,\cdots,L_{t-}^i,\cdots,L_t^m,X_t))dM_t^i \\
&+ U_t(\theta(t,T,L_t^0,\cdots,L_t^i,\cdots,L_t^m,X_t) - \theta(t,T,L_{t-}^0,\cdots,L_{t-}^i,\cdots,L_{t-}^m,X_t))dM_t^0 \\
&+ U_t\langle\boldsymbol{\theta},Q^*X_t\rangle dt + U_t\langle\boldsymbol{\theta},dM_t\rangle.
\end{aligned}$$

Note that $\overline{V}(t,T)$ is a bounded $\Im_t^L \vee \Im_t^X-$martingale. Consequently, we have

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \sum_{i=0}^m a^iL^i\frac{\partial}{\partial L^i} - \sum_{i=0}^m c^iL^i)\theta(t,T,L^0,\cdots,L^m,x) + \langle\boldsymbol{\theta},Q^*x\rangle \\
&+\sum_{i=1}^m\langle\boldsymbol{\mu}_i,x\rangle E\left[(\theta(t,T,L^0,L^1,\cdots,L^i+Y^i,\cdots,L^m,x) - \theta(t,T,L^0,L^1,\cdots,L^i,\cdots,L^m,x))\right] \\
&+\langle\boldsymbol{\mu}_0,x\rangle E\left[(\theta(t,T,L^0+Y^0,\cdots,L^m+Y^m,x) - \theta(t,T,L^0,\cdots,L^m,x))\right] = 0.
\end{aligned} \tag{3.3}$$

Due to the affine structure of $L_t^i$ for $i=0,1,\cdots,m$, we try the solution

$$\theta(t,T,L^0,\cdots,L^m,x) = e^{\sum_{i=0}^m B_i(t,T)L^i + C(t,T,x)}, \tag{3.4}$$

where the terminal conditions are given by

$$B_i(T,T) = -d^i, C(T,T,x) = 0.$$

Write $\overline{\mathbf{C}}(t,T) = (e^{C(t,T,e_1)},\cdots,e^{C(t,T,e_N)})^* \in \mathbf{R^N}$. Substituting the solution to $\theta$ given by (3.4) into (3.3) gives

$$\begin{aligned}
&\langle\overline{\mathbf{C}}(t,T),x\rangle\sum_{i=0}^m L^i(\frac{\partial B_i}{\partial t} - a^iB_i(t,T) - c^i) + \langle\frac{\partial\overline{\mathbf{C}}(t,T)}{\partial t},x\rangle \\
+\ \ &\langle\overline{\mathbf{C}}(t,T),Q^*x\rangle + \langle\overline{\mathbf{C}}(t,T),x\rangle\sum_{i=1}^m\langle\boldsymbol{\mu}_i,x\rangle\int_0^\infty(e^{B_i(t,T)y}-1)\langle\mathbf{f}^i(y),x\rangle dy \\
+\ \ &\langle\overline{\mathbf{C}}(t,T),x\rangle\langle\boldsymbol{\mu}_0,x\rangle(\prod_{i=0}^m\int_0^\infty e^{B_i(t,T)y}\langle\mathbf{f}^i(y),x\rangle dy - 1) = 0.
\end{aligned} \tag{3.5}$$

Since (3.5) holds for all $L^i$ and $x$, we have

$$\frac{\partial B_i}{\partial t} - a^i B_i(t,T) - c^i = 0, B_i(T,T) = -d^i, \qquad i = 0, 1, \cdots, m,$$

and

$$\frac{\partial \overline{\mathbf{C}}}{\partial t} + (Q + \mathbf{diag}(\overline{\mathbf{F}}_t))\overline{\mathbf{C}}(t,T) = 0, \overline{\mathbf{C}}(T,T) = \mathbf{1},$$

where $\overline{\mathbf{F}}_t$ is an $N$-dimensional vector with the $j$th component given by

$$\overline{F}_t^j = \sum_{i=1}^m \mu_i^j \int_0^\infty (e^{B_i(t,T)y} - 1) f^{ij}(y)dy + \mu_0^j (\prod_{i=0}^m \int_0^\infty e^{B_i(t,T)y} f^{ij}(y)dy - 1).$$

By solving the above equations, we complete the proof of (3.1). □

**Corollary 3.1.** *Let* $\mathbf{c} = (c^0, c^1, \cdots, c^m)^* \in \mathbf{R^{m+1}}$ *and* $\mathbf{d} = (d^0, d^1, \cdots, d^m)^* \in \mathbf{R^{m+1}}$ *with* $c^i \geq 0, d^i > 0$, *for each* $i = 0, 1, \cdots, m$. *Then, for* $k = 0, 1, \cdots, m$,

$$E\left[ L_T^k e^{-\int_t^T \sum_{i=0}^m c^i L_s^i ds - \sum_{i=0}^m d^i L_T^i} | \Im_t^L \vee \Im_t^X \right]$$

$$= e^{-\sum_{i=0}^m (c^i \xi^i(t,T) + d^i e^{-a^i(T-t)}) L_t^i} \langle e^{-a^k(T-t)} L_t^k \Psi_1(\mathbf{c}, \mathbf{d}, t, T) - \Psi_2^k(\mathbf{c}, \mathbf{d}, t, T), X_t \rangle, \tag{3.6}$$

*where* $\xi^i(t,T)$ *is given in Theorem 3.1, and*

$$\Psi_2^k(\mathbf{c}, \mathbf{d}, t, T) = \frac{\partial \Psi_1(\mathbf{c}, \mathbf{d}, t, T)}{\partial d^k}. \tag{3.7}$$

**Proof.** Differentiating both sides of (3.1) with respect to $d^k$ gives (3.6). □

**Corollary 3.2.** *For* $j \in \{0, 1, 2\}$, *let* $\hat{\mathbf{c}}_1 = (\hat{c}_1^0, \cdots, \hat{c}_1^m)^*$ *with* $\hat{c}_1^0 = 0$ *and* $\hat{c}_1^i = p_{ij}, i = 1, 2, \cdots, m$. *Then, the survival distribution for name* $j$ *is given by*

$$P(\tau_j > t) = e^{-\sum_{i=1}^m p_{ij} \lambda_0^i t} \langle \Psi_1(\hat{\mathbf{c}}_1, \mathbf{0}, 0, t), X_0 \rangle, \ t > 0.$$

*For* $i, j \in \{0, 1, 2\}, t > 0$, *let* $\hat{\mathbf{c}}_2 = (\hat{c}_2^0, \cdots, \hat{c}_2^m)^*$ *with* $\hat{c}_2^0 = 0$ *and* $\hat{c}_2^k = 1 - \overline{p}_{ki} \overline{p}_{kj}, k = 1, 2, \cdots, m$. *Then, we have*

$$P(\tau_i > t, \tau_j > t) = e^{-\sum_{k=1}^m \hat{c}_2^k \lambda_0^k t} \langle \Psi_1(\hat{\mathbf{c}}_2, \mathbf{0}, 0, t), X_0 \rangle, \ t > 0.$$

*Also, let* $\hat{\mathbf{c}}_3 = (\hat{c}_3^0, \cdots, \hat{c}_3^m)^*$ *with* $\hat{c}_3^0 = 0$ *and* $\hat{c}_3^i = 1 - \prod_{l=0}^2 \overline{p}_{il}, i = 1, 2, \cdots, m$. *Then, we have*

$$P(\tau_0 > t, \tau_1 > t, \tau_2 > t) = e^{-\sum_{k=1}^m \hat{c}_3^k \lambda_0^k t} \langle \Psi_1(\hat{\mathbf{c}}_3, \mathbf{0}, 0, t), X_0 \rangle,$$

*where* $\Psi_1(\hat{\mathbf{c}}_i, \mathbf{0}, 0, t) = \lim_{d^0, \cdots, d^m \to 0} \Psi_1(\hat{\mathbf{c}}_i, \mathbf{d}, 0, t)$ *with* $\Psi_1(\hat{\mathbf{c}}_i, \mathbf{d}, 0, t)$ *determined by (3.2).*

**Proof.** Since

$$P(\tau_j > t) = E\left[ e^{-\int_0^t \sum_{k=1}^m \lambda_s^k p_{kj} ds} \right] = e^{-\sum_{k=1}^m \lambda_0^k p_{kj} t} E\left[ e^{-\int_0^t \sum_{k=1}^m L_s^k p_{kj} ds} \right],$$

$$P(\tau_i > t, \tau_j > t) = E\left[e^{-\int_0^t \sum_{k=1}^{m} \lambda_s^k (1-\overline{p}_{ki}\overline{p}_{kj})ds}\right] = e^{-\sum_{k=1}^{m} \lambda_0^k (1-\overline{p}_{ki}\overline{p}_{kj})t} E\left[e^{-\int_0^t \sum_{k=1}^{m} L_s^k (1-\overline{p}_{ki}\overline{p}_{kj})ds}\right],$$

$$P(\tau_0 > t, \tau_1 > t, \tau_2 > t) = E\left[e^{-\int_0^t \sum_{k=1}^{m} \lambda_s^k (1-\prod_{l=0}^{2}\overline{p}_{kl})ds}\right] = e^{-\sum_{k=1}^{m} \lambda_0^k (1-\prod_{l=0}^{2}\overline{p}_{kl})t} E\left[e^{-\int_0^t \sum_{k=1}^{m} L_s^k (1-\prod_{l=0}^{2}\overline{p}_{kl})ds}\right],$$

an application of Theorem 3.1 yields the results. □

## 4. Markov copula model

Within the framework of Markov copula, the common shock model discussed in Section 2 was considered by Crépy et al. [6] and Bielecki et al. [1]. In this section, we present the corresponding Markov copula model.

Define the default process as

$$\mathbf{H}_t = (H_t^0, H_t^1, H_t^2) \in \{0,1\}^3,$$

where $H_t^i = 1_{\{\tau_i \le t\}}$. Then, $\mathbf{H}$ can be visualized as a finite state Markov chain and the state space $S$ of $H$ contains the following eight states:

state 1: (0,0,0), state 2: (1,0,0), state 3: (0,1,0), state 4: (0,0,1),

state 5: (1,1,0), state 6: (1,0,1), state 7: (0,1,1), state 8: (1,1,1).

Let $\psi = (X, L_t^0, L_t^1, \cdots, L_t^m)_{0<t\le T}$ be an $(m+2)$-dimensional stochastic process. Based on the default dependence constructed in Section 2, we now give the infinitesimal generator $\Lambda_{[\psi]}(t) = (\wedge_{ij}(t|\psi))_{8\times 8}$ for $\mathbf{H}$ given the path of $\psi$. Write state $i$ as $\mathbf{k} = (k_0, k_1, k_2)$ and state $j$ as $\mathbf{l} = (l_0, l_1, l_2)$. Note that $k_i \in \{0,1\}$ and $l_i \in \{0,1\}$, for $i = 0,1,2$. For notational convenience, we define

$$S_1 = \{i \in \{0,1,2\} : k_i = 1, l_i = 0\},$$

$$S_2 = \{i \in \{0,1,2\} : k_i = 0, l_i = 0\},$$

$$S_3 = \{i \in \{0,1,2\} : k_i = 0, l_i = 1\}.$$

Hence, for $\mathbf{k} \neq \mathbf{l}$, if $S_1$ is nonempty, then the transition intensity from state $i$ to state $j$ is

$$\wedge_{ij}(t|\psi) = 0.$$

If $S_1$ is empty, then

$$\wedge_{ij}(t|\psi) = \sum_{k=1}^{m} \lambda_t^k \prod_{i\in S_2} \overline{p}_{ki} \prod_{j\in S_3} p_{kj}.$$

For $\mathbf{k} = \mathbf{l}$, we have

$$\wedge_{ii}(t|\psi) = -\sum_{j=1, j\neq i}^{8} \wedge_{ij}(t|\psi).$$

For example, if $\mathbf{k} = (0,0,1), \mathbf{l} = (1,1,1)$, then $\wedge_{48}(t|\psi) = \sum_{i=1}^{m} \lambda_t^i p_{i1} p_{i2}$.

By using the forward Kolmogorov equation, the conditional transition probability matrix $P(t,u|\psi) = (P_{ij}(t,u|\psi))_{8\times 8}$ is governed by

$$\frac{dP(t,u|\psi)}{du} = P(t,u|\psi)\Lambda_{[\psi]}(t), \quad 0 \le t \le u,$$

with

$$P(t,t|\psi) = I.$$

The individual transition probability $P_{ij}(t,s|\psi)$ satisfies the following system of ODE:

$$\frac{dP_{ij}(t,u|\psi)}{du} = \sum_{k=1}^{8} P_{ik}(t,u|\psi) \wedge_{kj} (t|\psi), \quad 0 \le t \le u,$$

with

$$P_{ij}(t,t|\psi) = \begin{cases} 1, & i=j, \\ 0, & i \neq j. \end{cases}$$

Since $\Lambda_{[\psi]}(t)$ is upper triangular, individual transition probability $P_{ij}(t,u|\psi)$ can be solved successively in a sequential manner. Then, applying the expectation operator $E_\psi[.]$, which is the expectation taken over the path of $(X, L_t^0, L_t^1, \cdots, L_t^m)_{0<t\le T}$, we can obtain the transition probability $P_{ij}(t,u)$. Once these transition probabilities are available, the marginal distributions and the joint distribution of the default times can be derived.

For example, the conditional transition probability from state 1 to state 1, denote by $p_{11}$, can be obtained by solving the equation

$$\frac{dP_{11}(t,u|\psi)}{du} = -P_{11}(t,u|\psi) \sum_{k=1}^{m} \lambda_v^k (1 - \prod_{l=0}^{2} \overline{p}_{kl}),$$

with the boundary condition $P_{11}(t,t|\psi) = 1$. The solution to the above equation is given by

$$P_{11}(t,u|\psi) = e^{-\int_t^u \sum_{k=1}^{m} \lambda_v^k (1 - \prod_{l=0}^{2} \overline{p}_{kl}) dv}.$$

Therefore,

$$P(\tau_1 \wedge \tau_2 \wedge \tau_3 > t) = P_{11}(0,t) = E\left[ e^{-\int_0^t \sum_{k=1}^{m} \lambda_v^k (1 - \prod_{l=0}^{2} \overline{p}_{kl}) dv} \right].$$

Based on the connection between the dynamic Markov model $(\psi, H)$ and a common shock model, explicit formulas for the conditional and unconditional transition probabilities can be derived. Here, we omit the details of these formulas.

## 5. CDS and the first-to-default basket swap on three underlyings

In this section, we compute the fair spreads of a single-name credit default swap with and without the counterparty risk and the first-to-default basket swap on three underlyings when $L_t^i$ is modeled by (2.4) for $i = 0, 1, \cdots, m$.

### 5.1. Single-name credit default swap

In this subsection, we consider the impact of default risk of the protection seller on the spreads of a CDS. Specifically, we compute the fair credit default swap premium with and without default risk of the protection seller and the investor.

For simplicity, let the face value of the CDS be a monetary unit. It is assumed that the spread is paid continuously in time. Let $T$ be the maturity date of the CDS, $\kappa$ be the fair spread rate of a CDS contract without the default risk of the protection seller and the protection buyer, and $\kappa_1$ be the fair spread rate of a CDS contract with counterparty risk. Furthermore, if the protection seller defaults, then the protection buyer gets nothing. In the literature, much research has been carried out to study the impact of counterparty risk on CDS valuation. In this paper, the impact on the CDS spread rate in the presence of the counterparty risk measured by $\kappa_1 - \kappa$, has also been studied in Leung and Kwok [21].

We first describe the cash flows of a CDS without counterparty. For the default leg, the protection seller covers the credit losses $1 - R_1$ as soon as the reference entity has defaulted. For the premium leg, the protection buyer pays $\kappa$ to the seller continuously until maturity or until the reference entity defaults before maturity. Then, the fair spread of the CDS without counterparty risk is determined so that the discounted payoff of the two legs are equal when the contract is initiated at time 0. That is, the spread $\kappa$ should satisfy

$$\kappa \int_0^T E\left[1_{\{\tau_1 > u\}} D(0,u)\right] du = (1 - R_1) E\left[D(0,\tau_1) 1_{\{\tau_1 \le T\}}\right].$$

Hence,

$$\kappa = \frac{(1 - R_1) E\left[D(0,\tau_1) 1_{\{\tau_1 \le T\}}\right]}{\int_0^T E\left[1_{\{\tau_1 > u\}} D(0,u)\right] du}. \tag{5.1}$$

We now turn to the cash flows of a CDS with counterparty risk. For the default leg, if the reference entity defaults first before maturity, or the reference and the investor default simultaneously before maturity while the protection seller still survives, then the protection seller covers the credit losses $1-R_1$. For simplicity, we assume that if the protection seller or the buyer defaults first before maturity, then the protection buyer gets nothing. For the premium leg, the protection buyer pays $\kappa_1$ to the seller continuously until maturity or until any of names 0, 1, 2 defaults before maturity. Again, the fair spread of the CDS with counterparty risk is determined so that the discounted payoff of the two legs are equal when the contract is initiated at time 0. So, the spread $\kappa_1$ should satisfy

$$\kappa_1 \int_0^T E\left[1_{\{\tau_0 \wedge \tau_1 \wedge \tau_2 > u\}} D(0,u)\right] du = (1 - R_1) E\left[D(0,\tau_1)(1_{\{\tau_1 \le T, \tau_1 < \tau_2 \wedge \tau_0\}} + 1_{\{\tau_1 \le T, \tau_1 = \tau_0 < \tau_2\}}\right].$$

So,

$$\kappa_1 = \frac{(1 - R_1) E\left[D(0,\tau_1)(1_{\{\tau_1 \le T, \tau_1 < \tau_2 \wedge \tau_0\}} + 1_{\{\tau_1 \le T, \tau_1 = \tau_0 < \tau_2\}}\right]}{\int_0^T E\left[1_{\{\tau_0 \wedge \tau_1 \wedge \tau_2 > u\}} D(0,u)\right] du}. \tag{5.2}$$

**Proposition 5.1.** *Let $\overline{\mathbf{c}}_1 = (\overline{c}_1^0, \overline{c}_1^1, \cdots, \overline{c}_1^m)^*$, with $\overline{c}_1^0 = 1$ and $\overline{c}_1^i = p_{i1}$ for each $i = 1, \cdots, m$. Then, the fair CDS premium without counterparty risk is given by*

$$\kappa = \frac{(1 - R_1) \int_0^T e^{-(r_0 + \sum_{k=1}^m \lambda_0^k p_{k1}) u} \sum_{k=1}^m p_{k1} \langle \lambda_0^k \Psi_1(\overline{\mathbf{c}}_1, \mathbf{0}, 0, u) - \Psi_2^k(\overline{\mathbf{c}}_1, \mathbf{0}, 0, u), X_0 \rangle du}{\int_0^T e^{-(r_0 + \sum_{k=1}^m p_{k1} \lambda_0^k) t} \langle \Psi_1(\overline{\mathbf{c}}_1, \mathbf{0}, 0, t), X_0 \rangle dt}, \tag{5.3}$$

*where*

$$\Psi_1(\overline{\mathbf{c}}_1, \mathbf{0}, 0, u) = \lim_{d^0, \cdots, d^m \to 0} \Psi_1(\overline{\mathbf{c}}_1, \mathbf{d}, 0, u), \;\; \Psi_2^k(\overline{\mathbf{c}}_1, \mathbf{0}, 0, u) = \lim_{d^0, \cdots, d^m \to 0} \frac{\partial \Psi_1(\overline{\mathbf{c}}_1, \mathbf{d}, 0, u)}{\partial d^k},$$

*with $\Psi_1(\overline{\mathbf{c}}_1, \mathbf{d}, 0, u)$ determined by (3.2).*

**Proof.** The expected present value of the contingent payment paid by the protection seller from 0 to $T$ is given by

$$(1 - R_1) E\left[D(0,\tau_1) 1_{\{\tau_1 \le T\}}\right] = (1 - R_1) E\left[\int_0^T D(0,u) 1_{\{\tau_1 > u-\}} dH_u^1\right]$$

$$
\begin{aligned}
&= (1-R_1)E\left[\int_0^T e^{-\int_0^u (r_v+\sum\limits_{k=1}^m \lambda_v^k p_{k1})dv}\sum_{k=1}^m \lambda_u^k p_{k1}du\right]\\
&= (1-R_1)\int_0^T e^{-(r_0+\sum\limits_{k=1}^m \lambda_0^k p_{k1})u}\sum_{k=1}^m(\lambda_0^k+L_u^k)p_{k1}E\left[e^{-\int_0^u (L_v^0+\sum\limits_{k=1}^m L_v^k p_{k1})dv}\right]du\\
&= (1-R_1)\int_0^T e^{-(r_0+\sum\limits_{k=1}^m \lambda_0^k p_{k1})u}\sum_{k=1}^m p_{k1}\langle \lambda_0^k\Psi_1(\overline{\mathbf{c}}_1,\mathbf{0},0,u)-\Psi_2^k(\overline{\mathbf{c}}_1,\mathbf{0},0,u),X_0\rangle du,
\end{aligned}
$$

where the second equality is due to Lemma 2.1 and the fact that $H_t^1-\int_0^t 1_{\{\tau_1>u\}}q_1(u)du$ is an $\{\Im_t\}$-martingale, and the last equality follows from Theorem 3.1 and Corollary 3.1.

The total expected present value of the premium payment from 0 to $T$ is

$$
\begin{aligned}
\kappa\int_0^T E\left[1_{\{\tau_1>u\}}D(0,u)\right]du &= \kappa\int_0^T E\left[e^{-\int_0^u (r_v+\sum\limits_{k=1}^m \lambda_v^k p_{k1})dv}\right]du\\
&= \kappa\int_0^T e^{-(r_0+\sum\limits_{i=1}^m p_{i1}\lambda_0^i)t}\langle\Psi_1(\overline{\mathbf{c}}_1,\mathbf{0},0,t),X_0\rangle dt,
\end{aligned}
$$

where the first equality is obtained using Lemma 2.1, and the last equality follows from Theorem 3.1.

Finally, substituting the above expressions into (5.1) yields the result. □

**Proposition 5.2.** *Let $\overline{\mathbf{c}}_2=(\overline{c}_2^0,\overline{c}_2^1,\cdots,\overline{c}_2^m)^*$, with $\overline{c}_2^0=1$ and $\overline{c}_2^i=1-\prod_{l=0}^2\overline{p}_{il}$ for each $i=1,\cdots,m$. Then the fair CDS premium with counterparty risk is given by*

$$
\kappa_1=\frac{(1-R_1)\int_0^T e^{-(r_0+\sum\limits_{i=1}^m \overline{c}_2^i\lambda_0^i)t}\sum\limits_{k=1}^m p_{k1}\overline{p}_{k2}\langle\lambda_0^k\Psi_1(\overline{\mathbf{c}}_2,\mathbf{0},0,u)-\Psi_2^k(\overline{\mathbf{c}}_2,\mathbf{0},0,t),X_0\rangle dt}{\int_0^T e^{-(r_0+\sum\limits_{k=1}^m \overline{c}_2^k\lambda_0^k)t}\langle\Psi_1(\overline{\mathbf{c}}_2,\mathbf{0},0,t),X_0\rangle dt}, \tag{5.4}
$$

*where*

$$
\Psi_1(\overline{\mathbf{c}}_2,\mathbf{0},0,u)=\lim_{d^0,\cdots,d^m\to 0}\Psi_1(\overline{\mathbf{c}}_2,\mathbf{d},0,u),\ \Psi_2^k(\overline{\mathbf{c}}_2,\mathbf{0},0,u)=\lim_{d^0,\cdots,d^m\to 0}\frac{\Psi_1(\overline{\mathbf{c}}_1,\mathbf{d},0,u)}{\partial d^k},
$$

*with $\Psi_1(\overline{\mathbf{c}}_2,\mathbf{d},0,u)$ determined by (3.2).*

**Proof.** Let $\overline{\tau}=\min\limits_{s\in\mathcal{S}}\tau_s$. Then, by using Lemma 2.2, the expected present value of the contingent payment paid by the protection seller from 0 to $T$ is given by

$$
\begin{aligned}
&(1-R_1)(E\left[D(0,\tau_{\{1\}})1_{\{\overline{\tau}=\tau_{\{1\}}\le T\}}\right]+E\left[D(0,\tau_{\{0,1\}})1_{\{\overline{\tau}=\tau_{\{0,1\}}\le T\}}\right])\\
&= (1-R_1)\int_0^T E\left[e^{-\int_0^u (r_v+\sum\limits_{k=1}^m \lambda_v^k(1-\prod\limits_{l=0}^2\overline{p}_{kl}))dv}\sum_{k=1}^m\lambda_u^k p_{k1}\overline{p}_{k2}\right]du\\
&= (1-R_1)\int_0^T e^{-(r_0+\sum\limits_{i=1}^m(1-\prod\limits_{l=0}^2\overline{p}_{il})\lambda_0^i)t}\sum_{k=1}^m p_{k1}\overline{p}_{k2}\langle\lambda_0^k\Psi_1(\overline{\mathbf{c}}_2,\mathbf{0},0,t)-\Psi_2^k(\overline{\mathbf{c}}_2,\mathbf{0},0,t),X_0\rangle dt,
\end{aligned}
$$

where the last equality follows from Theorem 3.1 and Corollary 3.1.

The total expected present value of the premium payment from 0 to $T$ is

$$\kappa_1 \int_0^T E\left[1_{\{\overline{\tau}>u\}} D(0,u)\right] du = \kappa_1 \int_0^T E\left[ e^{-\int_0^u (r_v + \sum\limits_{k=1}^{m} \lambda_v^k (1-\prod\limits_{l=0}^{2} \overline{p}_{kl}) dv} \right] du$$

$$= \kappa_1 \int_0^T e^{-(r_0 + \sum\limits_{i=1}^{m} (1-\prod\limits_{l=0}^{2} \overline{p}_{il}) \lambda_0^i) t} \langle \Psi_1(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t), X_0 \rangle dt,$$

where the first equality holds because $\overline{\tau}$ has the $\Im$-intensity $\sum_{k=1}^m \lambda_v^k (1 - \prod_{l=0}^2 \overline{p}_{kl})$, and the last equality follows from Theorem 3.1.

Finally, substituting the above expressions into (5.2) yields the formula for the spread $\kappa_1$. □

### 5.2. First-to-default basket swap on three underlyings

A $k$th-to-default basket swap, which is a commonly traded product of portfolio credit derivatives, is a bilateral contract between an insurance buyer and an insurance seller. The payment streams of this derivative depend on the default times of an underlying portfolio of $n$ credit-risky assets. In this paper, we consider the first-to-default swap on three underlyings with maturity $T$. Assume that the default dependence structure of the three underlyings is the same as that of the investor, the reference entity, and the protection seller defined in the previous sections. Consider a unit notional and a constant recovery $R$. Let $\overline{\tau} = \tau_0 \wedge \tau_1 \wedge \tau_2$. In order to cover the loss when a credit event occurs, the buyer of protection pays a continuous premium (also called spread) till the first default occurs or till the maturity time $T$ of the contract if no default occurs before the maturity. Therefore, the fair spread of the first-to-default swap $c$ should satisfy

$$cE\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\overline{\tau}>t\}} dt\right] = (1-R) E\left[e^{-\int_0^{\overline{\tau}} r_s ds} 1_{\{\overline{\tau}\leq T\}}\right]. \quad (5.5)$$

From (5.5), we have the following result.

**Proposition 5.3.** *The fair spread of the first-to-default swap on the three underlyings is given by*

$$c = \frac{(1-R)\int_0^T e^{-(r_0 + \sum\limits_{k=1}^{m} \overline{c}_2^k \lambda_0^k) t} \sum\limits_{k=1}^{m} \overline{c}_2^k \langle \lambda_0^k \Psi_1(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t) - \Psi_2^k(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t), X_0 \rangle dt}{\int_0^T e^{-(r_0 + \sum\limits_{k=1}^{m} \overline{c}_2^k \lambda_0^k) t} \langle \Psi_1(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t), X_0 \rangle dt}, \quad (5.6)$$

*where* $\overline{\mathbf{c}}_2$, $\Psi_1(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t)$, *and* $\Psi_2^k(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t)$ *are defined in Proposition 5.2.*

**Proof.** Similar to the proof of Proposition 5.2, the left side of (5.5) is given by

$$c\int_0^T E\left[e^{-\int_0^t r_s ds} 1_{\{\overline{\tau}>t\}}\right] dt = c\int_0^T e^{-(r_0 + \sum\limits_{i=1}^{m} (1-\prod\limits_{l=0}^{2} \overline{p}_{il}) \lambda_0^i) t} \langle \Psi_1(\overline{\mathbf{c}}_2, \mathbf{0}, 0, t), X_0 \rangle dt.$$

The right side of (5.5) can be expressed as

$$(1-R) E\left[e^{-\int_0^{\overline{\tau}} r_u du} 1_{\{\overline{\tau}\leq T\}}\right] = (1-R) E\left[\sum_{s\in\mathcal{S}} e^{-\int_0^{\tau_s} r_u du} 1_{\{\overline{\tau}=\tau_s\leq T\}}\right]$$

$$= (1-R)\int_0^T E\left[ e^{-\int_0^u (r_v + \sum\limits_{k=1}^{m} \lambda_v^k (1-\prod\limits_{l=0}^{2} \overline{p}_{kl})) dv} \sum_{k=1}^{m} \lambda_u^k (1 - \prod_{i=0}^{2} \overline{p}_{ki}(u)) \right] du$$

$$= \quad (1-R)\int_0^T e^{-(r_0+\sum_{i=1}^{m}\bar{c}_2^i\lambda_0^i)t}\sum_{k=1}^{m}\bar{c}_2^k\langle\lambda_0^k\Psi_1(\bar{\mathbf{c}}_2,\mathbf{0},0,t)-\Psi_2^k(\bar{\mathbf{c}}_2,\mathbf{0},0,t),X_0\rangle dt.$$

Equating the above two equalities ends the proof. $\square$

## 6. Numerical results

In this section, we carry out a numerical study to examine the impact of some model parameters on the spreads of the CDS. Since the semi-analytic formulas for the spreads of the CDS have been obtained, we can calibrate the proposed model according to the structure of market data. Giesecke et al. [17] suggest there exist three regimes and obtain the transitional probability by making analysis on the corporate bond market over the course of the last 150 years. Therefore, the generator of the Markov chain can be borrowed from Giesecke et al. [17]. The groups of the shock events can be set as the cardinality of the set $\mathcal{S}$. Thus, the parameters $\eta = (\mathbf{r_0}, \boldsymbol{\lambda}_\mathbf{0}^{\boldsymbol{i}}, p_{ij}, \boldsymbol{\mu_i}, \mathbf{f}^i)$ for $i = 1, \cdots, m$ and $j = 0, 1, 2$ can be obtained according to

$$\eta = \mathop{\mathbf{arg\,min}}_{\hat{\eta}} \sum_{T\in\{T_1,\cdots,T_k\}} \frac{(\kappa(T,\hat{\eta})-\kappa(T))^2}{\kappa(T)^2},$$

where $T_1, \cdots, T_k$ are different maturities.

Since there are much more parameters to be estimated, we will work on the numerical calibrations in the future's research. In this section, we mainly perform a numerical analysis for CDS valuation. Following Giesecke et al. [17], we consider $N = 3$, that is, $X$ has three states, where state $e_1$, state $e_2$, and state $e_3$ represent a "good" economy, a "moderate" economy, and a "bad" economy, respectively. Also, the generator of the Markov chain can be borrowed from Giesecke et al. [17]. Let $m = 4$, $R_1 = R = 0.4$, $T = 5$, $\mathbf{r} = (0.05, 0.03, 0.01)^*$, $p_{1i} = 0.1$, $i = 0, 1, 2$, $p_{20} = 0.1$, $p_{31} = 0.1$, $p_{42} = 0.1$, $p_{21} = p_{22} = p_{30} = p_{32} = p_{40} = p_{41} = 0$, $\boldsymbol{\lambda}_\mathbf{0}^1 = (0.01, 0.03, 0.05)^*$, $\boldsymbol{\lambda}_\mathbf{0}^2 = (0.015, 0.045, 0.075)^*$, $\boldsymbol{\lambda}_\mathbf{0}^3 = (0.025, 0.075, 0.125)^*$, $\boldsymbol{\lambda}_\mathbf{0}^4 = (0.02, 0.06, 0.1)^*$, and $\boldsymbol{\mu_k} = (1, 3, 5)^*$ for $k = 1, 2, 3, 4$. For each $k = 0, 1, 2, 3, 4$, $\mathbf{f}^k$ is given by

$$\begin{cases} f^{k1}(x) &= 20e^{-20x}, x > 0, \\ f^{k2}(x) &= 10e^{-10x}, x > 0, \\ f^{k3}(x) &= 2e^{-2x}, x > 0. \end{cases}$$

Set $q_{12} = q_{21} = q_{23} = q_{32} = q_{13} = q_{31} = q$. To perform the numerical analysis, we use the fourth-order Runge-Kutta algorithm to solve (3.2) and use Simpson's 1/3 rule to calculate the integrations.

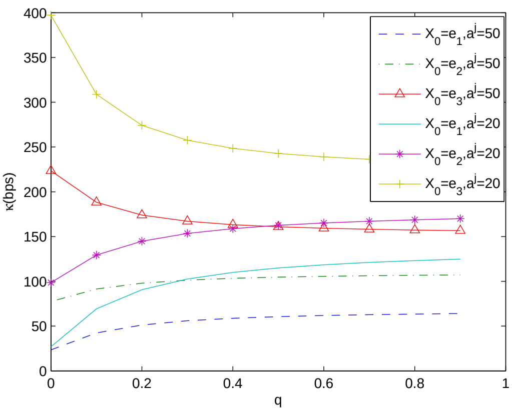



Figure 1: impact of $q$ on $\kappa$ for different $a^i$ and $X_0$, $\boldsymbol{\mu_0} = (1, 3, 5)^*$

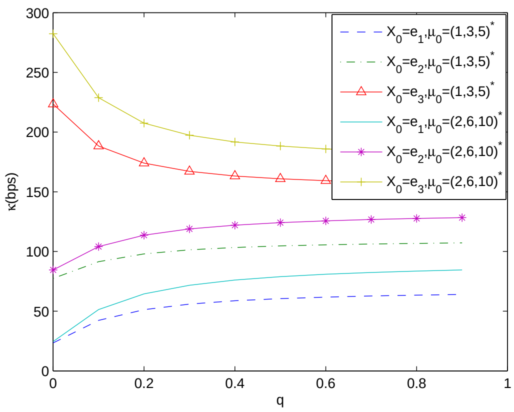
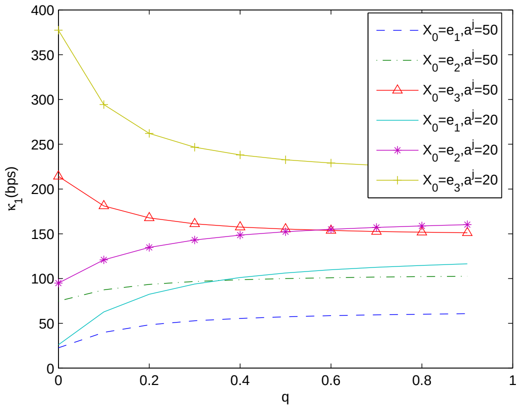



Figure 2: impact of $q$ on $\kappa$ for different $\boldsymbol{\mu_0}$ and $X_0$, $a^i = 50$



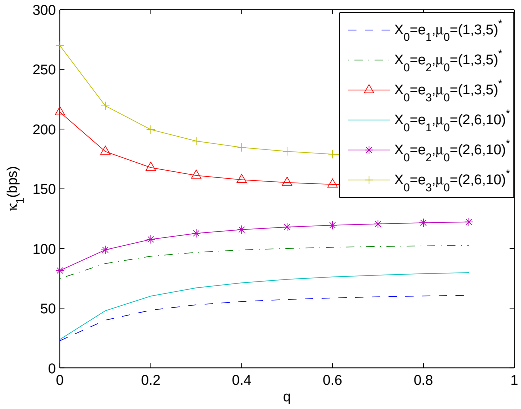
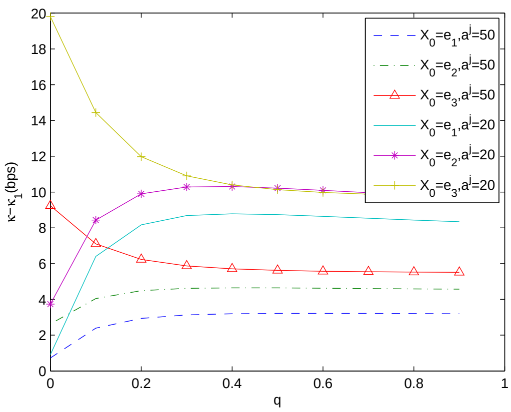




Figure 3: impact of $q$ on $\kappa_1$ for different $a^i$ and $X_0$, $\boldsymbol{\mu_0} = (1, 3, 5)^*$

Figure 4: impact of $q$ on $\kappa_1$ for different $\boldsymbol{\mu_0}$ and $X_0, a^i = 50$

Figures 1-2 present the impact of $q$ on the CDS spread without counterparty risk. In these figures, we see that the spread in the case with the "good" economy at time $t = 0$ is much lower. We also see that a larger $q$ results in a larger spread if $X_0 = e_1$ or $X_0 = e_2$. This is because the probability of switching to a worse economy increases as $q$ increases. On the other hand, if we start at the "bad" economy, the spreads decrease as $q$ increases. This is mainly due to the increasing probability of switching to a better economy. In Figure 1, we observe that the impact of the parameter $a^i$ on the spread $\kappa$ is very obvious with a larger $a^i$ corresponding to a lower spread. This may be explained by the fact that the time period that the intensity $\lambda^i$ goes back to the previous level of intensity immediately after major events occur will be shorten as $a^i$ increases. In Figure 2, we see that the spread increases with $\boldsymbol{\mu}_0$ with other parameters being fixed. Since an increase in $\boldsymbol{\mu}_0$ leads to a higher frequency that the intensities jump upward, the default probability for name 1 increases.

Figures 3-4 present the impact of $q$ on the CDS spread with counterparty risk. The curves in Figures 3-4 are similar to those in Figures 1-2. Figures 1-4 indicate that the spread with counterparty risk is lower than the one without counterparty risk. This is consistent with the financial intuition.



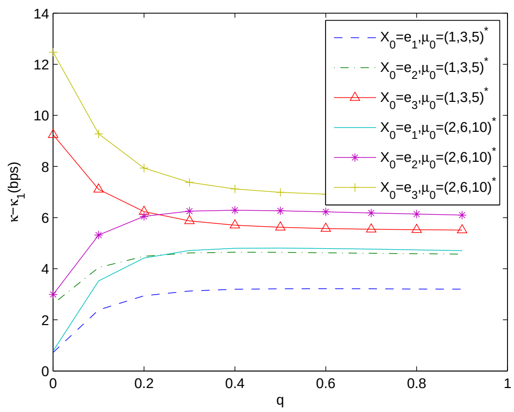




Figure 5: impact of $q$ on $\kappa - \kappa_1$ for different $a^i$ and $X_0$, $\boldsymbol{\mu_0} = (1, 3, 5)^*$

Figure 6: impact of $q$ on $\kappa - \kappa_1$ for different $\boldsymbol{\mu_0}$ and $X_0, a^i = 50$

Figures 5-6 present the impact of $q$ on the CDS spread difference $\kappa - \kappa_1$. For a fixed $a^i$, Figure 5 shows that the difference increases with $q$ when $X_0 = e_1$ or $X_0 = e_2$, while it decreases with $q$ when $X_0 = e_3$. We also see that a larger $\kappa - \kappa_1$ corresponds to a smaller $a^i$. In Figure 6, we observe that

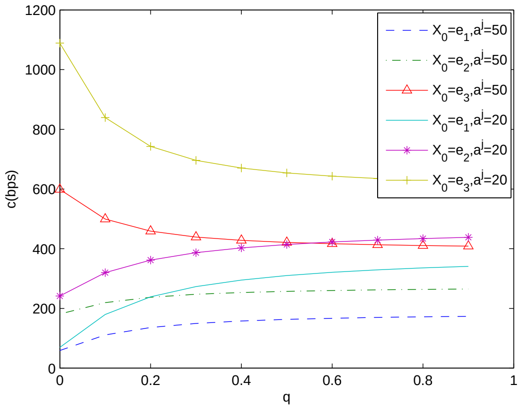



Figure 7: impact of $q$ on $c$ for different $a^i$ and $X_0$,$\boldsymbol{\mu_0} = (1, 3, 5)^*$

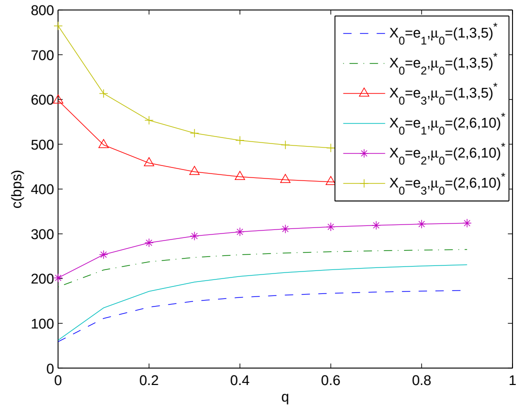



Figure 8: impact of $q$ on $c$ for different $\boldsymbol{\mu_0}$ and $X_0$, $a^i = 50$

the impact of the parameter $\boldsymbol{\mu}_0$ on $\kappa - \kappa_1$ is very obvious, and that the difference increases with $\boldsymbol{\mu}_0$.

Figures 7-8 present the impact of $q$ on the spread of the first-to-default basket swap on the three underlyings. The curves in Figures 7-8 are similar to those in Figures 1-4. Comparing Figures 7-8 with Figures 1-4, we see that the spread of the first-to-default basket swap is much higher than the single-name CDS spread. This is in line with the stylized feature: the first-to-default swap spread written on a portfolio of $n$ reference names increases with $n$.

## 7. Concluding remarks

In this paper, we use an intensity-based framework to analyze a CDS contract with counterparty credit risk. The proposed model is based on the idea that a firm's default is driven by idiosyncratic as well as other regional, sectoral, industry, or economy-wide shocks, whose arrivals are modeled by a multivariate regime-switching shot noise process. The regime-switching shot noise process can measure the impact on the intensities of major events well and allows us to obtain the joint Laplace transform of the regime-switching shot noise processes and the integrated regime-switching shot noise processes. Based on these formulas, we can calculate the CDS spread and the first-to-default swap spread.

The present work might be extended in at least two directions. A possible extension is that one can consider a contagion model with regime-switching shot noise intensities. Another possible extension is that the jump component $J_t^i$ in the intensities can be replaced by a more general process such as a Lévy subordinator with regime switching.

## Acknowledgements

The research of Yinghui Dong was supported by the NSF of Jiangsu Province (Grant No. BK20130260), the NNSF of China (Grant No. 11301369) and China Postdoctoral Science Foundation (Grant No. 2013M540371). The research of Kam C. Yuen was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7057/13P), and the CAE 2013 research grant from the Society of Actuaries - any opinions, finding, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the SOA. The research of Chongfeng Wu was supported by the NNSF of China (Grant No. 71320107002).

# References


1 Bielecki, T., S., Crépey, Jeanblanc, M., Zargari, B. 2012. Valuation and hedging of CDS counterparty exposure in a Markov copula model. *International Journal of Theoretical and Applied Finance*, 15 (1), 1-39.

2 Bielecki, T., S., Rutkowski, M. 2004. *Credit risk: modeling, valuation and hedging*. Springer.

3 Brigo, D., Pallavicini, A., Torresetti, R. 2010. Credit models and the crisis: default cluster dynamics and the generalized Poisson loss model, *Journal of Credit Risk*, 6(4): 39-81.

4 Buffington, J., Elliott, R. J. 2002. American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5: 497-514.

5 Cox, D. R. and Isham, V. 1986. The virtual waiting time and related processes, *Advance in Applied Probabality*, 18: 558-573.

6 Crépey, S., Jeanblanc, M., Zargari, B. 2009. Counterparty risk on a CDS in a Markov chain copula model with joint defaults, Working Paper.

7 Dassios, A., Jang, J. 2003. Pricing of catastrophe reinsurance & derivatives using the Cox process with shot noise intensity, *Finance and Stochastics*, 7(1),73-95.

8 Davis, M., V. Lo. 2001. Infectious defaults. *Quantitative Finance*, 1: 382-387.

9 Di Graziano, G., Rogers, L. C. G. 2009. A dynamic approach to the modelling of correlation credit derivatives using Markov chains. *International Journal of Theoretical and Applied Finance*, 12: 45-62.

10 Dong, Y., Yuen, K.C., Wu, C. 2014. Unilateral counterparty risk valuation of CDS using a regime-switching intensity model. *Statistics and Probability Letters,* 84, 23-35.

11 Duffie, D., Gârleanu, N. 2001. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, 57 (1): 41-59.

12 Duffie, D., Singleton, K. 1999. Modeling Term Structures of Defaultable Bonds. *Review of Financial Studies*, 12: 687-720.

13 Elliott R. J., Siu, T. K. 2011. Default times in a continuous-time Markovian regime switching model. *Stochastic Analysis and Applications*, 29 (5): 824-837.

14 Elliott, R.J., Aggoun, L., Moore, J. B. 1994. *Hidden Markov models: estimation and control*. Springer-Verlag: Berlin-Heidelberg-New York.

15 Gaspar, R. M., Schmidt, T. 2010. Credit risk modeling with shot-noise processes, working paper.

16 Giesecke, K. 2003. A simple exponential model for dependent defaults. *Journal of Fixed Income*, 13: 74-83.

17 Giesecke, K., Longstaff F. A., Schaefer S., Ilya Strebulaev I., 2011. Corporate bond default risk: A 150-year perspective. *Journal of Financial Economics*, 102, 233-250.

18 Hardy, M. R. 2001. A regime-switching model of long-term stock returns. *North American Actuarial Journal*, 5(2): 41-53.

19 Hull, J., White, A. 2004 Valuation of a CDO and a nth to default CDS without Monte Carlo simulation. *Journal of Derivatives*, 12: 8-23.

20 Jarrow, R., Turnbull, T. 1995. Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50(1): 53-85.

21 Leung, K. S., Kwok, Y. K. 2009. Counterparty risk for credit default swaps: Markov chain interacting intensities model with stochastic intensity. *Asia-Pacific Financial Markets*, 16: 169-181.

22 Lindskog, F., McNeil, A. 2003 Common Poisson shock models: applications to insurance and credit risk modelling. *Astin Bulletin*, 33: 209-238.

23 Ma, J., Yun, Y. Y. 2010. Correlated intensity, counter party risks, and dependent mortalities. *Insurance Mathematics and Economics*, 47: 337-351.

24 Schonbucher, P., Schubert, D. 2001. Copula dependent default risk in intensity models. Working Paper. Department of Statistics, Bonn University.

25 Shen, Y., Siu, T. K. 2013. Longevity bond Pricing under stochastic interest rate and mortality with regime switching. *Insurance: Mathematics and Economics*, 52(1): 114-123.

26 Yuen, F. L., Yang, H. L. 2009. Option pricing in a jump-diffusion model with regime-switching. *Astin Bulletin*, 39: 515-539.